A COUNTEREXAMPLE TO THE NON-SEPARABLE VERSION OF ROSENTHAL'S ℓ_1-THEOREM

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The following theorem is due to H. P. Rosenthal [6] and it provides a fundamental criterion for the embedding of ℓ_1 in Banach spaces.

Theorem 1 (Rosenthal's ℓ_1-theorem). *Let (x_n) be a bounded sequence in the Banach space X and suppose that (x_n) has no weakly Cauchy subsequence. Then (x_n) must contain a subsequence which is equivalent to the usual ℓ_1-basis.*

First of all, we recall that the sequence (x_n) is called weakly Cauchy if for each continuous functional $f \in X^*$, the scalar sequence (fx_n) is Cauchy. We also say that the sequence (x_n) is equivalent to the usual ℓ_1-basis if there are constants $A,\ B > 0$ such that for any $n \in \mathbb{N}$ and any scalars $a_1, a_2, \ldots, a_n$,

$$A\sum_{i=1}^{n}|a_i| \leq \|\sum_{i=1}^{n}a_i x_i\| \leq B\sum_{i=1}^{n}|a_i|.$$

The above condition guarantees that the linear map $T : \ell_1 \to \overline{\text{span}}\{x_n \mid n \in \mathbb{N}\}$, defined by $Te_n = x_n$ for any $n \in \mathbb{N}$, is an isomorphism and therefore the space ℓ_1 embeds in X.

A satisfactory extension of Theorem 1 to spaces of the type $\ell_1(\kappa)$, for κ an uncountable cardinal, would be desirable, since it would provide a useful criterion for the embedding of $\ell_1(\kappa)$ in Banach spaces. Consequently, R. Haydon [4] posed the following problem: Let κ be an uncountable cardinal. Suppose that X is a Banach space and A is a bounded subset of X with $\text{card}(A) = \kappa$, such that A does not contain any weakly Cauchy sequence. Can we deduce that A has a subset equivalent to the usual basis of $\ell_1(\kappa)$?

Before posing the question, Haydon [3] exhibited a counterexample for the case where the cardinal κ is equal to ω_1. A completely different counterexample, for the case of ω_1, was also obtained by J. Hagler [2]. Finally, a complete solution to this problem was given by C. Gryllakis [1] who proved that the answer is always negative with only one exception, namely when κ and $\text{cf}(\kappa)$ are both strong limit cardinals. However, Gryllakis' proof is quite difficult and, unlike the case of ω_1, does not construct any specific counterexample.

In what follows, our aim is to present a counterexample to the non-separable version of Rosenthal's ℓ_1-theorem and to give a complete answer to Haydon's problem. More precisely, for any uncountable cardinal κ, we construct a non-separable analogue of the Hagler Tree space (see [2]). In the case where either κ or $\text{cf}(\kappa)$ is not a strong limit cardinal, using the aforementioned construction, we obtain a Banach space X and a bounded subset A of X with $\text{card}(A) = \kappa$ such that (1) A contains no weakly Cauchy sequence and (2) no subset of A is equivalent to the usual $\ell_1(\kappa)$-basis. In the case where κ and $\text{cf}(\kappa)$ are both strong limit cardinals, the answer to Haydon's problem is positive (see [1]).

In the following we fix an infinite cardinal κ and we set

$$\begin{aligned}\{0,1\}^\kappa &= \Big\{a : \{\xi < \kappa\} \to \{0,1\}\Big\}\\ &= \Big\{(a_\xi)_{\xi<\kappa} \mid a_\xi = 0 \text{ or } 1\Big\}\\ \mathcal{D} = \{0,1\}^{<\kappa} &= \bigcup\Big\{\{0,1\}^\eta \mid \mathrm{Ord}(\eta), \eta < \kappa\Big\}\\ &= \Big\{(a_\xi)_{\xi<\eta} \mid \eta \text{ is an ordinal number, } \eta < \kappa,\ a_\xi = 0 \text{ or } 1\Big\}.\end{aligned}$$

The set $\mathcal{D}$ is called the *standard tree* and its elements are called *nodes*. The elemens of the set $\{0,1\}^\kappa$ are called *branches*.

If s is a node and $s \in \{0,1\}^\eta$ we say that s is on the *η-th level of $\mathcal{D}$* and we denote the level of s by $\mathrm{lev}(s)$. The initial segment partial ordering, denoted by $\leq$, is defined as follows: if $s = (a_\xi)_{\xi<\eta_1}$ and $s' = (b_\xi)_{\xi<\eta_2}$ belong to $\mathcal{D}$ then $s \leq s'$ if and only if $\eta_1 \leq \eta_2$ and $a_\xi = b_\xi$ for any $\xi < \eta_1$.

A linearly ordered subset $\mathcal{I}$ of $\mathcal{D}$ is called a *segment* if for every $s < t < s'$, t belongs to $\mathcal{I}$ provided that s, s' belong to $\mathcal{I}$. Consider now a non-empty segment $\mathcal{I}$ and let η_1 be the least ordinal number such that there is a node s with $\mathrm{lev}(s) = \eta_1$ and $s \in \mathcal{I}$. Moreover, suppose that there are an ordinal number η and a node s' on the η-th level such that $s \leq s'$ for any $s \in \mathcal{I}$. Let η_2 be the least ordinal satisfying this property. Then we say that $\mathcal{I}$ is an *η_1-η_2 segment*.

A finite family $\{\mathcal{I}_j\}_{j=1}^r$ of segments is called *admissible* if the following properties are satisfied

(1) there exist ordinals $\eta_1 < \eta_2$ such that each $\mathcal{I}_j$ is an η_1-η_2 segment,
(2) $\mathcal{I}_i \cap \mathcal{I}_j = \emptyset$ provided that $i \neq j$.

We next consider the vector space $c_{00}(\mathcal{D})$ of finitely supported functions $x : \mathcal{D} \to \mathbb{R}$. For a segment $\mathcal{I}$ of $\mathcal{D}$, we set $\mathcal{I}^*(x) = \sum_{s\in\mathcal{I}} x(s)$. Then, for any $x \in c_{00}(\mathcal{D})$ we define the norm

$$\|x\| = \sup\Big[\sum_{j=1}^r |\mathcal{I}_j^*(x)|^2\Big]^{1/2}$$

where the supremum is taken over all admissible families $\{\mathcal{I}_j\}_{j=1}^r$ of segments. We set X_κ the completion of $c_{00}(\mathcal{D})$ under this norm.

Now let $B = (a_\xi)_{\xi<\kappa}$ be any branch. Then B can be naturally identified with a maximal segment of $\mathcal{D}$, namely

$$B = \{s_0 < s_1 < \ldots < s_\eta < \ldots\}$$

where $s_0 = \emptyset$ and $s_\eta = (a_\xi)_{\xi<\eta}$. For any function $x \in c_{00}(\mathcal{D})$ we have already defined $B^*(x) = \sum_{s\in B} x(s)$. Clearly, $B^* : c_{00}(\mathcal{D}) \to \mathbb{R}$ is a linear functional of norm 1. This functional can be extended to a bounded functional on X_κ, which is denoted again by B^*. Let Γ be the set which contains the functionals B^* defined above. Clearly, Γ is a bounded subset of X_κ^* with $\mathrm{card}(\Gamma) = 2^\kappa$.

Concerning the space X_κ and the family of functionals Γ, we prove the following theorems.

Theorem 2. *Any sequence $(B_n^*)_{n\in\mathbb{N}}$ in Γ has a subsequence equivalent to the usual ℓ_1-basis. Therefore, Γ contains no weakly Cauchy sequence.*

Theorem 3. *No subset of Γ is equivalent to the usual basis of $\ell_1(\kappa^+)$.*

Now let κ be a cardinal number, which is not strong limit. This means that there exists cardinal $\lambda < \kappa$ such that $\kappa \leq 2^\lambda$. Consider the space X_λ and the corresponding family $\Gamma \subset X_\lambda^*$. Then we have $\mathrm{card}(\Gamma) = 2^\lambda$ and hence we can choose a subset A of Γ with $\mathrm{card}(A) = \kappa$. By Theorem 2, the set A contains no weakly Cauchy sequence. Furthermore, by Theorem 3, no subset of A is equivalent to the usual $\ell_1(\kappa)$-basis.

Moreover, in the case where κ is strong limit and $\mathrm{cf}(\kappa)$ is not a strong limit cardinal, using our construction, we obtain a Banach space X and a subset A of X with the desired properties.

Finally, the main properties of the spaces Hagler Tree [2] and James Tree [5], by which our construction is inspired, suggest the following conjecture for the spaces X_κ.

Conjecture. *The space X_κ does not contain a subspace isomorphic to $\ell_1(\kappa)$.*

Concerning the above conjecture, a partial result can be proved rather easily. For any node $s \in \mathcal{D}$, let $e_s \in X_\kappa$ be defined by $e_s(t) = 1$ if $t = s$ and $e_s(t) = 0$ otherwise. Now consider any branch B and the subspace $\overline{\mathrm{span}}\{e_s \mid s \in B\}$. Then this subspace contains no isomorphic copy of $\ell_1(\kappa)$.

References

[1] C. Gryllakis, *On the non-separable version of the ℓ_1-theorem of Rosenthal*, Bull. London Math. Soc. **18** (1987), 253-258.

[2] J. Hagler. *A counterexample to several questions about Banach spaces.* Studia Math. **60** (1977), 289-308.

[3] R. G. Haydon, *On Banach spaces which contain $\ell_1(\tau)$ and types of measures on compact spaces*, Israel J. Math. **28** (1977), 313-324.

[4] R. G. Haydon, *Non-separable Banach spaces*, Functional Analysis: surveys and recent results II (ed. K. D. Bierstedt and B. Fuchssteiner, North-Holland, Amsterdam, 1980), 19-30.

[5] R. C. James, *A separable somewhat reflexive Banach space with nonseparable dual*, Bull. Amer. Math. Soc. **80** (1974),738-743.

[6] H. P. Rosenthal, *A characterization of Banach spaces containing ℓ^1*, Proc. Nat. Acad. Sci. **71** (1974), 2411-2413.

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